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Analysis of Laminated Plates by Trigonometric Theory, Radial Basis, and Unified Formulation

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Nomenclature

a	=	side of plate
E_1, E_2	=	Young's moduli in material reference system
G_{12}, G_{13}, G_{23}	=	shear moduli in material reference system
h	=	thickness of plate
k	=	layer index
$\bar{N}$	=	dimensionless buckling load
$\bar{N}_{xx}, \bar{N}_{yy}, \bar{N}_{xy}$	=	applied in-plane loadings for buckling analysis
P	=	amplitude of sinusoidal transversal pressure load
$\bar{w}$	=	dimensionless transversal displacement
ν_{12}	=	Poisson's ratio in material reference system
ρ	=	density
$\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\tau}_{xz}, \bar{\tau}_{xy}$	=	dimensionless stress components
$\bar{\omega}$	=	normalized fundamental frequency

I. Introduction

THE analysis of thin and moderately thick plates has been modeled by thin-plate theories or by shear deformation theories. Typically, such theories involve a constant transverse displacement across the thickness direction, but this assumption is adequate only for thin plates. The use of shear deformation theories considering the contribution of the transverse normal strain and stress is fundamental for thick plates.

Among such theories, higher-order theories in the thickness direction were addressed by Noor [1], Lo et al. [2], Kant et al. [3], Librescu et al. [4], and Reddy [5], and more recent contributions on bending, vibration, and buckling of laminated and metallic plates can be found in books by Leissa [6] and Kapania and Raciti [7,8], and in a recent review on laminated composite shell vibration by Qatu et al. [9]. In particular, the work of Carrera [10,11] shows interesting ways of computing transverse and normal stresses in laminated composite or sandwich plates. The unified formulation (UF) proposed by Carrera, also known as CUF, is a powerful framework for the analysis of beams, plates, and shells, because it permits us to obtain the governing equations, irrespective of the shear deformation theory being considered. This formulation has been applied in several finite element analyses.

In this Note, we propose to use this UF to derive the equations of motion and boundary conditions to analyze isotropic and laminated plates by radial basis function (RBF) collocation, according to a sinus-based shear deformation theory that accounts for through-the-thickness deformations. This theory is an expansion of the developments by Dau et al. [12] and Vidal and Polit [13]. It considers through-the-thickness deformations, allowing for quadratic variation of the transverse displacement.

Recently, RBFs have enjoyed considerable success and research as a technique for interpolating data and functions. There has been an increased interest in their use for solving partial differential equations (PDEs). This approach, which approximates the whole solution of the PDE directly using RBFs, is truly a mesh-free technique. Kansa [14] introduced the concept of solving PDEs by an unsymmetric RBF collocation method based upon the multiquadric interpolation functions in which the shape parameter may vary across the problem domain. The use of alternative methods to the finite element methods for the analysis of plates is attractive due to the absence of a mesh and the ease of collocation methods. Ferreira has recently applied the RBF collocation to the static deformations of composite beams and plates [15,16].

In this Note, it is investigated how the UF can be combined with RBFs to the analysis of thick isotropic and laminated plates, using a sinus-based shear deformation theory that accounts for through-the-thickness deformations. The quality of the present method in predicting static deformations, free vibrations, and buckling loads of thick isotropic and laminated plates is compared and discussed with other methods in some numerical examples.

II. Governing Equations and Boundary Conditions

According to the CUF, the displacement $\boldsymbol{u} = \{u, v, w\}$ can be modeled as

$$\boldsymbol{u}(x, y, z) = F_\tau(z)\boldsymbol{u}_\tau(x, y) \tag{1}$$

where F_τ are the thickness functions, and they change depending on the considered theory. The variable τ (s for the variation $\delta\boldsymbol{u}$) is a sum index, and it varies from zero to the order of expansion n in the thickness direction.

In this work, the principle of virtual displacements is used to obtain the equations of motions and boundary conditions. In the general case of multilayered plates subjected to mechanical and inertial loads, the governing equations in strong form for the RBF method are

$$\delta\mathbf{u}_s^{kT}\colon\ \mathbf{K}_{uu}^{k\tau s}\mathbf{u}_\tau^k = \mathbf{M}^{k\tau s}\ddot{\mathbf{u}}_\tau^k + \mathbf{P}_{u\tau}^k \tag{2}$$

where T indicates the transpose, k indicates the layer, and double dots indicate the acceleration. $\mathbf{K}_{uu}^{k\tau s}$, $\mathbf{M}^{k\tau s}$, and $\mathbf{P}_{u\tau}^k$ are the fundamental nuclei for the elastic, inertial, and mechanical terms, respectively. In

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the static analysis, one can neglect the inertial term, while in the free vibration analysis, one can neglect the mechanical term.

The corresponding Neumann-type boundary conditions are

$$\Pi_d^{k\tau s}\mathbf{u}_\tau^k = \Pi_d^{k\tau s}\bar{\mathbf{u}}_\tau^k \tag{3}$$

where $\Pi_d^{k\tau s}$ is the fundamental nucleus for the boundary conditions, and the overline indicates an assigned condition.

The buckling analysis considers the addition into the equations of motion of terms

$$\bar{N}_{xx}\frac{\partial^2 w}{\partial x^2} + 2\bar{N}_{xy}\frac{\partial^2 w}{\partial x \partial y} + \bar{N}_{yy}\frac{\partial^2 w}{\partial y^2}$$

associated with the equation in w_0 being $\bar{N}_{xx}$, $\bar{N}_{xy}$, and $\bar{N}_{yy}$ the in-plane applied forces. To determine the critical buckling load of the laminated plate, the mechanical load and all inertial terms are set to zero.

For more details about fundamental nuclei and the governing equations, one can refer to [17].

The present sinus shear deformation theory is used to perform the analysis,

$$\begin{aligned} u &= u_0 + zu_1 + \sin\left(\frac{\pi z}{h}\right)u_3 \\ v &= v_0 + zv_1 + \sin\left(\frac{\pi z}{h}\right)v_3 \\ w &= w_0 + z^2 w_2 \end{aligned} \tag{4}$$

where u_0, v_0, and w_0 are translations of a point at the middle surface of the plate, w_2 is higher-order translations, and u_1, v_1, u_3, and v_3 denote rotations. This theory is an expansion of early developments by Vidal and Polit [13]. It considers a quadratic variation of the transverse displacement w, allowing for through-the-thickness deformations.

According to the RBF method, we consider a linear elliptic partial differential operator L and a bounded region Ω in $\mathbb{R}^n$ with some boundary $\partial\Omega$. In the static problems, we seek the computation of displacements $\mathbf{u}$ from the global system of equations:

$$\mathcal{L}\,\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \tag{5}$$

$$\mathcal{L}_B\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \tag{6}$$

where $\mathcal{L}$ and $\mathcal{L}_B$ are linear operators in the domain and on the boundary, respectively. The right-hand sides of Eqs. (5) and (6) represent the external forces applied on the plate and the boundary conditions applied along the perimeter of the plate, respectively.

The eigenproblem looks for eigenvalues λ and eigenvectors $\mathbf{u}$ that satisfy

$$\mathcal{L}\,\mathbf{u} + \lambda\mathbf{u} = 0 \quad \text{in } \Omega \tag{7}$$

$$\mathcal{L}_B\mathbf{u} = 0 \quad \text{on } \partial\Omega \tag{8}$$

The RBF ϕ approximation of a function $\mathbf{u}$ is given by

$$\tilde{\mathbf{u}}(\mathbf{x}) = \sum_{i=1}^{N}\alpha_i\phi(\|x - y_i\|_2), \qquad \mathbf{x} \in \mathbb{R}^n \tag{9}$$

where y_i, $i = 1, \ldots, N$, is a finite set of distinct points (centers) in $\mathbb{R}^n$. The coefficients α_i are chosen so that $\tilde{\mathbf{u}}$ satisfies some boundary conditions. Considering N distinct interpolations, and knowing $u(x_j)$, $j = 1, 2, \ldots, N$, we find α_i by the solution of a $N \times N$ linear system:

$$\mathbf{A}\,\underline{\alpha} = \mathbf{u} \tag{10}$$

where $\mathbf{A} = [\phi(\|x - y_i\|_2)]_{N\times N}$, $\underline{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$, and $\mathbf{u} = [u(x_1), u(x_2), \ldots, u(x_N)]^T$.

For the solution of the static problem and the eigenproblem, one can see [17].

III. Numerical Examples

All numerical examples consider a Chebyshev grid and a Wendland function, defined as

$$\phi(r) = (1 - cr)_+^8[32(cr)^3 + 25(cr)^2 + 8cr + 1] \tag{11}$$

where r is the Eucledian distance, and c the shape parameter obtained by an optimization procedure, as in Ferreira and Fasshauer [18].

A simply supported square laminated plate of side a and thickness h, composed of four equally layers oriented at [0/90/90/0°], is considered.

In the static case, the plate is subjected to a sinusoidal vertical pressure on the midplane of the form $p_z = P\sin(\pi x/a)\sin(\pi y/a)$, and the orthotropic material properties for each layer are given by $E_1 = 25.0E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, and $\nu_{12} = 0.25$. The in-plane displacements, the transverse displacements, the normal stresses, and the in-plane and transverse shear stresses are presented in normalized form as $\bar{w} = 10^2 w_{(a/2,a/2,0)}h^3E_2/Pa^4$, $\bar{\sigma}_{xx} = \sigma_{xx(a/2,a/2,h/2)}h^2/Pa^2$, $\bar{\sigma}_{yy} = \sigma_{yy(a/2,a/2,h/4)}h^2/Pa^2$, $\bar{\tau}_{xz} = \tau_{xz(0,a/2,0)}h/Pa$, and $\bar{\tau}_{xy} = \tau_{xy(0,0,h/2)}h^2/Pa^2$.

In Table 1, we present results for the present sinusoidal (SINUS) theory, using 13×13 up to 21×21 points. We compare results with higher-order solutions by Reddy [19], first-order shear deformation (FSDT) solutions by Reddy and Chao [20], and an exact solution by Pagano [21]. Our SINUS theory produces excellent results when compared with other higher-order shear deformation theories (HSDTs) for all h/a ratios and for transverse displacements, normal stresses, and transverse shear stresses. It is clear that the FSDT cannot be used for thick laminates.

In the free vibration problem, the material parameters of a layer are $E_1/E_2 = 10$, 20, 30, or 40; $G_{12} = G_{13} = 0.6E_2$, $G_3 = 0.5E_2$, $\nu_{12} = 0.25$, and $\rho = 1.\text{kg/m}^3$. The thickness-to-span ratio $h/a = 0.2$ is employed in the computation. Table 2 lists the fundamental frequencies of the simply supported laminate. It is found that the present meshless results are in very close agreement with the mesh-free results of Liew et al. [22], based on the FSDT, and the values of

Table 1 [0/90/90/0°] square laminated plate under sinusoidal load-SINUS formulation ($\epsilon_z \neq 0$)

h/a	Method	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\tau}_{zx}$	$\bar{\tau}_{xy}$
0.1	HSDT [19]	0.7147	0.5456	0.3888	0.2640	0.0268
	FSDT [20]	0.6628	0.4989	0.3615	0.1667	0.0241
	Three-dimensional elasticity [21]	0.743	0.559	0.403	0.301	0.0276
	Present (13 × 13 grid)	0.7257	0.5642	0.3949	0.2722	0.0275
	Present (17 × 17 grid)	0.7257	0.5642	0.3949	0.2825	0.0275
	Present (21 × 21 grid)	0.7257	0.5642	0.3949	0.2874	0.0275
0.01	HSDT [19]	0.4343	0.5387	0.2708	0.2897	0.0213
	FSDT [20]	0.4337	0.5382	0.2705	0.1780	0.0213
	Three-dimensional elasticity [21]	0.4347	0.539	0.271	0.339	0.0214
	Present (13 × 13 grid)	0.4371	0.5434	0.2740	0.3026	0.0215
	Present (17 × 17 grid)	0.4372	0.5438	0.2734	0.3146	0.0215
	Present (21 × 21 grid)	0.4373	0.5438	0.2734	0.3202	0.0215

Table 2 Normalized fundamental frequency of simply supported cross-ply laminated square plate [0/90/90/0°] [$\bar{\omega} = (\omega a^2/h)\sqrt{\rho/E_2}$ and $h/a = 0.2$]

		E_1/E_2			
Method	Grid	10	20	30	40
Liew et al. [22]		8.2924	9.5613	10.320	10.849
Exact (Khdeir and Librescu) [23]		8.2982	9.5671	10.326	10.854
Present SINUS	13 × 13	8.2725	9.5214	10.2541	10.7535
	17 × 17	8.2724	9.5213	10.2540	10.7534
	21 × 21	8.2724	9.5213	10.2540	10.7534

Table 3 Biaxial buckling load of three-layer [0/90/0°] simply supported laminated plate [$\bar{N} = \bar{N}_{xx}a^2/(E_2h^3)$, $\bar{N}_{xy} = 0$, and $\bar{N}_{yy} = \bar{N}_{xx}$]

Grid	SS	SC	CC
13 × 13	10.1287	11.5707	13.3459
17 × 17	10.1276	11.5696	13.3449
21 × 21	10.1274	11.5695	13.3448
Liew and Huang [24]	10.178	11.575	13.260
Khdeir and Librescu [23]	10.202	11.602	13.290

[23]. The relative errors between the analytical and present solutions are around 0.93% when we use a 13 × 13 grid for $E_1/E_2 = 10$, and they are 0.1% when we use a 13 × 13 grid for $E_1/E_2 = 40$.

In the buckling analysis, the span-to-thickness ratio h/a is taken to be 0.1 and the material properties are $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, and $\nu_{12} = 0.25$. Table 3 tabulates the biaxial buckling loads of a [0/90/0°] laminated plate with equal layers. The laminated plate is simply supported along the edges parallel to the x axis, while the other two edges may be simply supported (S) or clamped (C). Table 3 shows that an excellent agreement is achieved for all edge conditions when comparing the results obtained by the present RBF approach with those of Liew and Huang [24], who used a moving least-squares differential quadrature method approach, and the FSDT solutions by [23]. In Fig. 1, the first four buckling modes are shown for a biaxial bucking load of a three-layer [0/90/0°] SSSC laminated plate ($\bar{N} = \bar{N}_{xx}a^2/(E_2h^3)$, $\bar{N}_{xy} = 0$, and $\bar{N}_{yy} = \bar{N}_{xx}$), using a grid of 17 × 17 points.

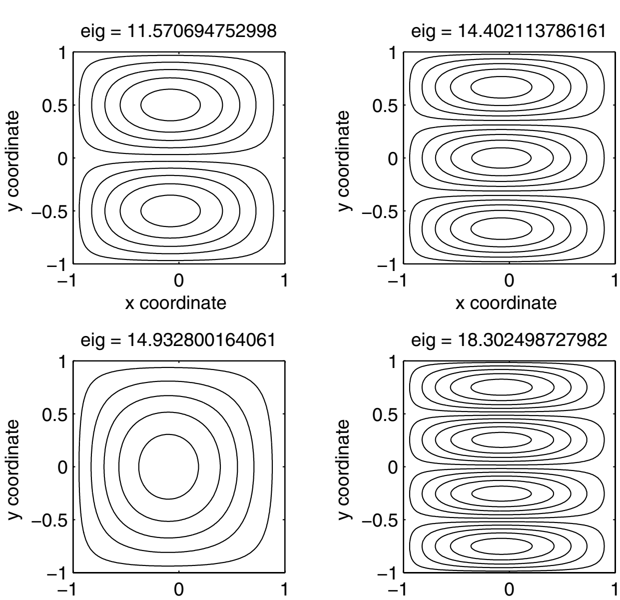

Fig. 1 First four buckling modes: biaxial buckling load of three-layer [0/90/0°] SSSC laminated plate [$\bar{N} = \bar{N}_{xx}a^2/(E_2h^3)$, $\bar{N}_{xy} = 0$, and $\bar{N}_{yy} = \bar{N}_{xx}$], grid 17 × 17 points.

IV. Conclusions

In this Note, a study using the RBF collocation method to analyze static deformations, free vibrations, and buckling loads of square cross-ply laminated plates using a sinus-based shear and normal deformation theory is presented, allowing for through-the-thickness deformations. Using the UF, all the C^o plate formulations can be easily discretized by RBF collocation. The present results were compared with existing analytical solutions or competitive finite element solutions, and excellent agreement was observed in all cases. The present method is a simple yet powerful alternative to other finite element or meshless methods.

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